

L^p ESTIMATES FOR THE HOMOGENIZATION OF STOKES PROBLEM IN A PERFORATED DOMAIN

AMINA MECHERBET & MATTHIEU HILLAIRET

ABSTRACT. In this paper, we consider the Stokes equations in a perforated domain. When the number of holes increases while their radius tends to 0, it is proven in [6], under suitable dilution assumptions, that the solution is well-approximated asymptotically by solving a Stokes-Brinkman equation. We provide here quantitative estimates in L^p -norms of this convergence.

1. INTRODUCTION

Let Ω be a connected smooth bounded domain in \mathbb{R}^3 . Given $N \in \mathbb{N}$, we consider $(B_i^N)_{i \in \{1, \dots, N\}}$ a family of N balls in \mathbb{R}^3 such that:

$$B_i^N := B\left(x_i^N, \frac{r_i^N}{N}\right) \subset \Omega, \text{ for all } i \in \{1, \dots, N\}.$$

Defining the perforated set \mathcal{F}^N by

$$\mathcal{F}^N = \Omega \setminus \bigcup_{i=1}^N B_i^N,$$

we denote $(u^N, \pi^N) \in H^1(\mathcal{F}^N) \times L_0^2(\mathcal{F}^N)$ (here the subscript 0 fixes that π^N has mean 0 on \mathcal{F}^N) the unique solution to the Stokes problem:

$$(1) \quad \begin{cases} -\Delta u^N + \nabla \pi^N &= 0, \\ \operatorname{div} u^N &= 0, \end{cases} \quad \text{on } \mathcal{F}^N,$$

completed with boundary conditions:

$$(2) \quad \begin{cases} u^N(x) &= V_i^N, & \text{on } \partial B_i^N, \\ u^N(x) &= 0, & \text{on } \partial \Omega, \end{cases}$$

where $(V_i^N)_{i=1, \dots, N} \in (\mathbb{R}^3)^N$ are given. In [6], the authors show that, if $r_i^N = 1$ uniformly, if the holes are sufficiently dilute and the empirical measures associated to the distributions of $(x_i^N, V_i^N)_{i=1, \dots, N}$ converge to a sufficiently smooth particle distribution function $f(x, v)dx dv$, then the associated sequence of velocity-fields $(u^N)_{N \in \mathbb{N}}$ converges weakly to

the velocity-field \bar{u} of the unique solution $(\bar{u}, \bar{\pi}) \in H^1(\Omega) \times L_0^2(\Omega)$ to the Stokes-Brinkman problem:

$$(3) \quad \begin{cases} -\Delta \bar{u} + \nabla \bar{\pi} &= (j - \rho \bar{u}), \\ \operatorname{div} \bar{u} &= 0, \end{cases} \quad \text{on } \Omega,$$

completed with boundary condition:

$$(4) \quad \bar{u} = 0 \quad \text{on } \partial\Omega.$$

In (3), the flux j and density ρ are computed respectively to the given particle distribution function f by:

$$j(x) = 6\pi \int_{\mathbb{R}^3} v f(x, v) dv \quad \rho(x) = 6\pi \int_{\mathbb{R}^3} f(x, v) dv, \quad \forall x \in \Omega.$$

We emphasize that here and below (in the definition of discrete densities and fluxes), we include the factor 6π in the formulas. This factor is reminiscent of the Stokes law for the resistance of a viscous fluid on a moving sphere (see next section). Via a standard compact-embedding argument, it entails from [6] that we have also strong convergence of the u^N to \bar{u} in L^p -spaces (for $p < 6$) up to the extraction of a subsequence. We are interested herein in providing a quantitative estimate of the convergence of u^N to \bar{u} .

This problem is related to the homogenization of Stokes problem in perforated domains with homogeneous boundary conditions and a forcing term. In this case, previous studies prove convergence of the sequence of N -hole solutions to the solution of the Stokes-Brinkman problem (or other ones depending on the dilution regime of the holes) in the periodic as in the random setting [1, 3, 14]. These results extend to the Stokes problem previous analysis for the Laplace equations [2]. The problem with non-homogeneous boundary conditions that we consider herein is introduced by [6] in a tentative to justify a Vlasov-Navier-Stokes or Vlasov-Stokes problem that is applied in spray theory [5, 8]. The strategy here is to couple the Stokes problem (1)-(2) by prescribing that the holes are particles whose position/velocity $(x_i^N, V_i^N)_{i=1, \dots, N}$ evolve according to Newton laws:

$$(5) \quad \frac{d}{dt} x_i^N = V_i^N,$$

$$(6) \quad m \frac{d}{dt} V_i^N = - \int_{\partial B_i^N} (\nabla u + \nabla u^\top - p \mathbb{I}_3) n d\sigma.$$

Here we denote by m the mass of the particles and n the normal to ∂B_i^N directed toward B_i^N . Note that, contrary to the stationary problem we are studying in this paper, in this target system the holes/particles are moving. As classical in these "many-particle systems", one crucial issue to complete a rigorous derivation is to control the distance between the particles. Partial improvements have been obtained in this direction either by increasing the family of datas for which transition from the N -hole stationary Stokes problem to the Stokes-Brinkman problem hold [11] or by completing successfully the kinetic program for the odes (5)-(6) with singular forcing terms [9]. In this paper, we do not tackle this issue on the distance between particles. Keeping in mind that, in the full problem, one wants to

couple the dynamical equations for the particles with the pde governing the fluid problem, we infer that a quantitative description of the convergence of the N -hole solutions to the solutions to the Stokes-Brinkman problem is necessary. So, we discuss in which norms such quantitative estimates may be computed.

We make precise now the main assumptions that are in force throughout the paper:

- the balls are sufficiently spaced:

$$(H1) \quad \exists C_0 > 0 \text{ independent of } i \neq j, N \text{ s.t. } \text{dist}(B_i^N, B_j^N) \geq \frac{C_0}{N^{\frac{1}{3}}}, \quad \text{dist}(B_i^N, \partial\Omega) \geq \frac{C_0}{N^{\frac{1}{3}}};$$

- the normalized radii $r_i^N > 0$ are uniformly bounded:

$$(H2) \quad \exists R_0 > 0 \text{ independent of } i, N \text{ s.t. } r_i^N \leq R_0;$$

- the kinetic energies of the data are uniformly bounded:

$$(H3) \quad \exists E_0 > 0 \text{ independent of } N \text{ such that } \frac{1}{N} \sum_{i=1}^N |V_i^N|^2 \leq |E_0|^2.$$

Then, following [6] and [11] we introduce empirical measures to describe the asymptotic behavior of the distribution $(x_i^N, V_i^N, r_i^N)_{i=1, \dots, N}$:

$$S_N(x, v, r) = \frac{1}{N} \sum_{i=1}^N \delta_{x_i^N, V_i^N, r_i^N}(x, v, r) \in \mathcal{P}(\mathbb{R}^3 \times \mathbb{R}^3 \times]0, \infty[).$$

We denote then by ρ^N and j^N its two first momentums:

$$\rho^N := 6\pi \int_{\mathbb{R}^3 \times]0, \infty[} S_N(dv dr), \quad j^N := 6\pi \int_{\mathbb{R}^3 \times]0, \infty[} v S_N(dv dr).$$

The sequence of densities ρ^N (resp. fluxes j^N) are then measures (resp. vectorial measures) on \mathbb{R}^3 with support in Ω . Compared to [6], the main new assumption is that the radii of the holes may depend on i, N . We restrict to the dilution regime of this previous reference for simplicity though it is likely that the result extends to the one of [11].

With the above assumptions, for arbitrary $N \in \mathbb{N}$, the domain \mathcal{F}^N has a smooth boundary and there exists a solution to (1)-(2) (see [7, Section IV]). We have thus at-hand a sequence $(u^N, \pi^N) \in H^1(\mathcal{F}^N) \times L_0^2(\mathcal{F}^N)$. Under assumption (H1)-(H2)-(H3) one may prove that up to the extraction of a subsequence ρ^N (resp. j^N) converges to some density $\rho \in L^\infty(\Omega)$ (resp. flux $j \in L^2(\Omega)$). We have then a unique solution $(\bar{u}, \bar{\pi})$ to the Stokes-Brinkman problem (3)-(4) for this density/flux pair (see next section for more details). In order to compute the distance between u^N and \bar{u} we extend u^N to the whole Ω by setting:

$$E_\Omega[u^N] := \begin{cases} u^N, & \text{on } \mathcal{F}^N, \\ V_j^N, & \text{on } B_j^N. \end{cases}$$

Because of boundary conditions (2), these extended velocity-fields satisfy $E_\Omega[u^N] \in H_0^1(\Omega)$. With these notations, we state now our two results on the convergence of the sequence $(E_\Omega[u^N])_{N \in \mathbb{N}}$ towards \bar{u} .

Theorem 1.1. *Assume that $j \in L^q(\Omega)$ for some $q > 3$ and $p \in]1, \frac{3}{2}[$. If R_0/C_0^3 is sufficiently small, there exists a constant $K > 0$ depending only on R_0, C_0, p, q, Ω for which:*

$$\|E_\Omega[u^N] - \bar{u}\|_{L^p(\Omega)} \leq K \left[\|j^N - j\|_{(C^{0,1}(\bar{\Omega}))^*} + \|\rho^N - \rho\|_{(C^{0,1}(\bar{\Omega}))^*} + \frac{\|j\|_{L^q(\Omega)} + E_0}{N^{1/3}} \right],$$

for $N \geq (4R_0/C_0)^{3/2}$.

Theorem 1.2. *Given $p \in]1, \frac{3}{2}[$ there exists $K > 0$ depending only on $R_0, C_0, p, \|\rho\|_{L^\infty(\Omega)}, \Omega$ for which:*

$$\|E_\Omega[u^N] - \bar{u}\|_{L^p(\Omega)} \leq K \left[\|j - j^N\|_{(C^{0,1}(\bar{\Omega}))^*} + \left(\|\rho - \rho^N\|_{(C^{0,1}(\bar{\Omega}))^*} + \frac{1}{N^{1/3}} \right)^{1/3} E_0 \right],$$

for $N \geq (4R_0/C_0)^{3/2}$.

The two previous theorems give a quantitative estimate of the weak-convergence obtained in [6]. They link the convergence of the sequence $(u^N)_{N \in \mathbb{N}}$ to \bar{u} to the convergence of the fluxes and densities in the so-called bounded-lipschitz or Fortet-Mourier distance (see [15, Section 6]). As the $(\rho^N)_{N \in \mathbb{N}}$ are positive measures on Ω with the same finite mass, we may relate the bounded-lipschitz distance $\|\rho - \rho^N\|_{(C^{0,1}(\bar{\Omega}))^*}$ to the Wasserstein distance between ρ^N and ρ thanks to the Kantorovich-Rubinstein formula [15, Theorem 5.10]. The restriction on the values N is irrelevant as our aim is to describe the asymptotics $N \rightarrow \infty$ of u^N . It is due to the fact that our method requires that $B(x_j^N, r_j^N/N) \subset B(x_j^N, C_0/4N^{1/3})$ for arbitrary $j \in \{1, \dots, N\}$.

The results we state are complementary one to the other. The first one is limited to sufficiently small ratios R_0/C_0^3 . This can be interpreted as configurations for which the holes are sufficiently small compared to their relative distances. In this case, the convergence estimate is linear with respect to the convergence of the data ρ^N and j^N . The second result is valid for arbitrary data. The counterpart is that the convergence estimate is now sublinear with respect to the convergence of the densities ρ^N . These results can be extended in several directions. First, we may interpolate these convergences with crude uniform bounds on $E_\Omega[u^N]$ in $L^6(\Omega)$ to extend the convergence to L^p spaces with $p \geq 3/2$. But we can also generalize the result by considering convergence of the empirical measures in more general dual spaces. We comment at the end of the paper on the estimates we can attain with this method.

The outline of the paper is as follows. In the next section, we state and prove some technical lemmas on the resolution of the Stokes problem and Stokes-Brinkman problem. In particular, we state a regularity lemma in negative Sobolev spaces which is at the heart of our computations. Section 3 is devoted to the proof of our main results and we provide a discussion on the possible extensions of our results in a closing section.

We list below some possible non-standard notations that we use during the proofs. First, we use extensively localizing procedures around the balls B_j^N so that we use repeatedly the shortcut $A(x, r_{int}, r_{ext})$ for the annulus with center x and internal (resp. external) radius r_{int} (resp. r_{ext}). We also use the notations $\oint_A u$ for the mean of u on the set of positive measure A :

$$\oint_A u(x)dx = \frac{1}{|A|} \int_A u(x)dx.$$

We denote classically $L^p(\Omega)$ (resp. $W^{m,p}(\Omega)$ or $H^m(\Omega)$) Lebesgue spaces (resp. Sobolev spaces) on Ω . The index zero specifies zero mean when added to Lebesgue spaces and vanishing boundary-values when added to Sobolev spaces. For instance, we denote:

$$L_0^2(\Omega) := \left\{ v \in L^2(\Omega), \oint_{\Omega} v = 0 \right\}, \quad D_0(\Omega) := \{ v \in [H_0^1(\Omega)]^3, \operatorname{div} v = 0 \}.$$

When there is no ambiguity concerning the definition domain, we only use exponents to denote norms:

$$\| \cdot \|_q := \| \cdot \|_{L^q(\Omega)}, \quad \| \cdot \|_{m,q} := \| \cdot \|_{W^{m,q}(\Omega)}.$$

Given $\alpha \in (0, 1]$ and $\Omega \subset \mathbb{R}^3$, we also introduce $\mathcal{C}^{0,\alpha}(\bar{\Omega})$, the set of α -Hölder continuous functions on $\bar{\Omega}$. When Ω is bounded, this is a Banach space endowed with the norm:

$$\|f\|_{\mathcal{C}^{0,\alpha}(\bar{\Omega})} = \|f\|_{\infty} + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^{\alpha}}.$$

Given an arbitrary smooth domain Ω and $q \in (1, \infty)$, we denote $\mathfrak{B} : L_0^q(\Omega) \rightarrow W_0^{1,q}(\Omega)$ the so-called Bogovskii operator (see [7, Section III.3]). It is a continuous linear map which, given $f \in L_0^q(\Omega)$ provides a solution u to the problem:

$$\begin{cases} \operatorname{div} u &= f, & \text{on } \Omega, \\ u &= 0, & \text{on } \partial\Omega. \end{cases}$$

If $\Omega = A(x_0, r_{int}, r_{ext})$, we specify the Bogovskii operator by indices: $\mathfrak{B}_{x_0, r_{int}, r_{ext}}$. Such operators have been extensively studied in [1]. The main results we apply here are summarized in [11, Appendix A].

Finally, in the whole paper we use the symbol \lesssim to express an inequality with a multiplicative constant depending on irrelevant parameters.

2. PRELIMINARY RESULTS ON THE STOKES AND STOKES-BRINKMAN EQUATIONS

In this section, we prove some lemmas concerning the resolution of the Stokes and Stokes-Brinkman equations that will help in the the proofs of our main results.

2.1. Analysis of the Stokes-Brinkman equation in a bounded domain. In this whole part Ω is a fixed smooth bounded domain. Given a boundary condition $u^* \in H^{1/2}(\Omega)$ and $\rho \in L^\infty(\Omega)$, we consider the Stokes-Brinkman problem:

$$(7) \quad \begin{cases} \rho u - \Delta u + \nabla \pi &= j, \\ \operatorname{div} u &= 0, \end{cases} \quad \text{on } \Omega,$$

completed with boundary condition:

$$(8) \quad u = u^* \quad \text{on } \Omega.$$

We assume below that $\rho \geq 0$ including possibly $\rho = 0$. In this latter case, the Stokes-Brinkman equations degenerate into the Stokes equations. We refer the reader to [7, Section IV] for a comprehensive study of Stokes equations. Herein, we also apply the variational characterization of solutions that is provided in [11, Section 2]. It is straightforward to extend the existence theory of these references to the Stokes-Brinkman equations with an arbitrary bounded weight $\rho \geq 0$ yielding the following theorem:

Theorem 2.1. *Let $j \in L^{6/5}(\Omega; \mathbb{R}^3)$ and $\rho \in L^\infty(\Omega)$ such that $\rho \geq 0$. Given $u^* \in H^{1/2}(\Omega)$ satisfying:*

$$\int_{\partial\Omega} u^* \cdot n d\sigma = 0,$$

the following equivalent statements hold true and furnish a solution to (7)-(8):

- i) *There exists a unique pair $(u, \pi) \in H^1(\Omega) \times L_0^2(\Omega)$ satisfying (7) in the sense of $\mathcal{D}'(\Omega)$ and (8) in the sense of traces;*
- ii) *There exists a unique divergence-free $u \in H^1(\Omega)$ satisfying (8) in the sense of traces and:*

$$(9) \quad \int_{\Omega} \nabla u : \nabla v = \int_{\Omega} (j - \rho u) \cdot v, \quad \text{for all } v \in D_0(\Omega);$$

- iii) *if we assume furthermore that $j = 0$, there exists a unique solution to the minimisation problem:*

$$(10) \quad \inf \left\{ \frac{1}{2} \int_{\Omega} |\nabla v|^2 + \rho |v|^2, v \in [H^1(\Omega)]^3, \operatorname{div} v = 0, v = u^* \text{ on } \partial\Omega \right\}.$$

The proof of this theorem is a straightforward extension of [7, Section IV] and [11, Section 2] and is left to the reader.

As stated in [7, Theorem IV.6.1], in the case $\rho = 0$ and $u^* = 0$ we have also that, if $j \in W^{m,p}(\Omega)$ for some $m \in \mathbb{N}$ and $p \in (1, \infty)$ then the solution u satisfies $u \in W^{m+2,p}(\Omega)$. We may extend this regularity statement to our Stokes-Brinkman problem:

Theorem 2.2. *Let $\rho \in L^\infty(\Omega)$ such that $\rho \geq 0$ and assume that $u^* = 0$, $j \in L^q(\Omega)$, for some $q \in [6/5, \infty)$. Then, there exists a unique pair $(u, \pi) \in W^{2,q}(\Omega) \times W^{1,q}(\Omega)$ satisfying (7)-(8). Moreover, there exists $C = C(\Omega, q, \|\rho\|_\infty) > 0$ such that:*

$$\|u\|_{2,q} \leq C \|j\|_q.$$

Proof. Because Ω is bounded and $q \geq 6/5$ we have that $j \in L^{6/5}(\Omega)$. Theorem 2.1 yields the existence and uniqueness of the solution $(u, \pi) \in H^1(\Omega) \times L_0^2(\Omega)$. We recall that we focus on homogenous boundary conditions. In this case $u \in H_0^1(\Omega)$ so that Poincaré inequality entails that $\|u\|_{1,2} \lesssim \|\nabla u\|_2$.

At first, let assume further that $q \leq 6$. Because $H_0^1(\Omega) \subset L^q(\Omega)$, we remark that (u, π) satisfies the Stokes equation with data $f = j - \rho u \in L^q(\Omega)$. The regularity theorem for Stokes equations implies that $(u, \pi) \in W^{2,q}(\Omega) \times W^{1,q}(\Omega)$ with:

$$\|u\|_{2,q} \leq C\|j - \rho u\|_q \leq C(\|j\|_q + \|\rho\|_\infty \|u\|_q)$$

for some positive constant $C > 0$ depending only on Ω and q . Thus, we want to bound $\|u\|_q$ by $\|j\|_q$. To this end, we apply the weak-formulation of Stokes-Brinkman problem (9) with $v = u \in D_0(\Omega)$ to get that:

$$\begin{aligned} \int_{\Omega} |\nabla u|^2 &= \int_{\Omega} j \cdot u - \int_{\Omega} \rho |u|^2 \\ &\leq \|j\|_q \|u\|_{q'} \\ &\lesssim \|j\|_q \|\nabla u\|_2 \end{aligned}$$

where we applied again the embedding $H_0^1(\Omega) \subset L^{q'}(\Omega)$ since $q \geq 6/5$. This entails that $\|u\|_q \lesssim \|u\|_{1,2} \lesssim \|j\|_2$ and concludes the proof.

If we assume now $q > 6$ we iterate the same argument. Indeed, because Ω is bounded, we have in particular that $j \in L^6(\Omega)$ so that the previous reasoning applies yielding:

$$\|u\|_{2,6} \leq C\|j\|_{L^6(\Omega)} \leq C'\|j\|_{L^q(\Omega)}.$$

We may then apply the continuous embedding $W^{2,6}(\Omega) \subset W^{1,\infty}(\Omega)$. Hence, we obtain now again that $j - \rho u \in L^q(\Omega)$ and we conclude by application of the regularity theorem for Stokes equations as previously. \square

Keeping in mind that we want to compare the N -solution $E_\Omega[u^N]$ with \bar{u} on Ω , we do not expect to be able to use a regular theory for the Stokes (or Stokes-Brinkman) equations as above. Indeed, the u^N are solutions to the Stokes equations on \mathcal{F}^N only. Even if we were extending the pressure π^N to $E_\Omega[\pi^N]$ by fixing a constant on the $B(x_i^N, r_i^N/N)$ (say 0 for instance), we expect that $\Delta E_\Omega[u^N] - \nabla E_\Omega[\pi^N]$ contains single layer distributions on the interfaces fluid/holes. Fortunately, these single layer distributions are regular enough to compute L^p -estimates as depicted below. These L^p -estimates are adapted from weak-regularity statements for stationary Stokes equations that have been obtained in the study of fluid-structure interaction problems [13, Appendix 1].

Given $p \in]1, 6[$, we introduce the following norm of $v \in H_0^1(\Omega)$:

$$[v]_{p,\Omega} := \sup \left\{ \left| \int_{\Omega} \nabla v : \nabla w \right|, w \in W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega), \operatorname{div} w = 0, \|w\|_{2,p'} = 1 \right\}.$$

We have then:

Lemma 2.3. *Let $p \in]1, 6[$. There exists a non-negative $C = C(\Omega, p)$ such that:*

$$\|v\|_p \leq C[v]_{p,\Omega},$$

for all divergence-free $v \in H_0^1(\Omega)$.

Similary, we define the following norm based on the weak-formulation for the Stokes-Brinkman equations. Given $\rho \in L^\infty(\Omega)$ such that $\rho \geq 0$, we set:

$$[v]_{p,\Omega,\rho} := \sup \left\{ \left| \int_{\Omega} \nabla v : \nabla w + \int_{\Omega} \rho v \cdot w \right|, \right. \\ \left. w \in W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega), \operatorname{div} w = 0, \|w\|_{2,p'} = 1 \right\}$$

Then, there holds:

Lemma 2.4. *Let $p \in]1, 6[$ and $\rho \in L^\infty(\Omega)$ such that $\rho \geq 0$. There exists a non-negative $C = C(\Omega, p, \|\rho\|_\infty)$ that satisfies:*

$$\|v\|_p \leq C[v]_{p,\Omega,\rho},$$

for all divergence-free $v \in H_0^1(\Omega)$.

As Lemma 2.3 can be obtained by setting $\rho = 0$ in Lemma 2.4, we prove only the second one.

Proof. The idea is to use the following equality:

$$\|v\|_p = \sup \left\{ \left| \int_{\Omega} v \cdot \phi \right|, \quad \phi \in L^{p'}(\Omega) \quad \|\phi\|_{L^{p'}(\Omega)} = 1 \right\}.$$

Let $p \leq 6$ and $\phi \in L^{p'}(\Omega)$, $\|\phi\|_{p'} = 1$. Because $p' \geq 6/5$, we introduce the unique solution (u_ϕ, π_ϕ) to the problem

$$(11) \quad \begin{cases} -\Delta u_\phi + \nabla \pi_\phi + \rho u_\phi &= \phi, \\ \operatorname{div} u_\phi &= 0, \end{cases} \quad \text{on } \Omega,$$

completed with the boundary condition $u_\phi = 0$ on $\partial\Omega$. According to Lemma 2.2, this solution satisfies $u_\phi \in W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega)$, $\pi_\phi \in W^{1,p'}(\Omega)$ and

$$\|u_\phi\|_{2,p'} \leq C\|\phi\|_{p'} \leq C.$$

Moreover, we have that $W^{2,p'}(\Omega) \subset W^{1,2}(\Omega)$. This yields that, using an integration by parts:

$$\begin{aligned} \int_{\Omega} v \cdot \phi &= \int_{\Omega} v \cdot (-\Delta u_\phi + \nabla \pi_\phi + \rho u_\phi) \\ &= \int_{\Omega} \nabla u_\phi : \nabla v + \int_{\Omega} \rho u_\phi \cdot v. \end{aligned}$$

This entails:

$$\left| \int_{\Omega} v \cdot \phi \right| \leq [v]_{p,\Omega,\rho} \|u_{\phi}\|_{2,p'} \leq C[v]_{p,\Omega,\rho}.$$

We obtain:

$$\left| \int_{\Omega} v \cdot \phi \right| \leq C[v]_{p,\Omega,\rho}, \quad \forall \phi \in L^{p'}(\Omega), \|\phi\|_{p'} = 1.$$

□

2.2. The Stokes problem in an exterior domain. In this part, we focus on the case $\Omega = \mathbb{R}^3 \setminus B(0, r)$ where $r > 0$. Given $V \in \mathbb{R}^3$, we consider the Stokes problem on Ω :

$$(12) \quad \begin{cases} -\Delta u + \nabla \pi &= 0, \\ \operatorname{div} u &= 0, \end{cases} \quad \text{on } \Omega,$$

completed with boundary conditions:

$$(13) \quad u(x) = V, \text{ on } \partial B(0, r), \quad \lim_{|x| \rightarrow \infty} |u(x)| = 0.$$

We investigate here the convergence of Stokes solutions on annuli to the Stokes solution on the exterior domain. Precisely, let $R > r$ and $\Omega_R = B(0, R) \setminus \overline{B(0, r)} = A(0, r, R)$. We denote by (u_R, π_R) the solution to:

$$(14) \quad \begin{cases} -\Delta u_R + \nabla \pi_R &= 0, \\ \operatorname{div} u_R &= 0, \end{cases} \quad \text{on } \Omega_R,$$

completed with boundary conditions:

$$(15) \quad u(x) = V, \text{ on } \partial B(0, r), \quad u(x) = 0, \text{ on } \partial B(0, R).$$

We emphasize that we only consider constant boundary conditions. In this particular case existence theory for (12)-(13) is well known since explicit formulas for the solutions are part of the folklore (see [12] and more recently [6]). Explicit solutions for (14)-(15) are also available following the same construction scheme as in the unbounded case. We refer here to [6, Section 6.2] for more details. On the basis of these formulas, the convergence of (u_R, π_R) to (u, π) is studied in [6]. For later purpose, we complement here this study with two supplementary properties of this convergence.

First, we denote

$$F_R^r = \int_{\partial B(0,r)} (\nabla u_R - \pi_R \mathbb{I}) n d\sigma, \quad F^r = \int_{\partial B(0,r)} (\nabla u - \pi \mathbb{I}) n d\sigma.$$

We use the symbol \mathbb{I} here for the identity matrix in \mathbb{R}^3 . These quantities are related to the force exerted by the flow (u_R, π_R) (resp. (u, π)) on the hole $B(0, R)$ (see Appendix A for more details). We recall that Stokes law states that $F^r = 6\pi r V$. The following lemma shows that the sequence F_R^r converges to F^r . Moreover, explicit formulas for u_R and u allow to compute the rate of this convergence:

Lemma 2.5. *There holds:*

$$|F_R^r - F^r| \lesssim r^2 \frac{|V|}{R}.$$

Proof. We show the inequality for $r = 1$. The result extends to any $r > 0$ by a standard scaling argument that we recall afterwards.

We have that:

$$F_R^1 - F^1 = \int_{\partial B(0,1)} (\nabla(u_R - u)) n d\sigma + \int_{\partial B(0,1)} (\pi - \pi_R) n d\sigma.$$

Adopting the notations introduced in [6] we set $r = |x|$, $\omega = \frac{x}{|x|}$ and $P_\omega V = (\omega \cdot V)\omega$. We have then, for arbitrary $x \in A(0, 1, R)$ that:

$$\begin{aligned} u_R(x) = & - \left[4A(R)r^2 + 2B(R) + \frac{C(R)}{r} - \frac{D(R)}{r^3} \right] (\mathbb{I} - P_\omega)V \\ & - 2 \left[A(R)r^2 + B(R) + \frac{C(R)}{r} + \frac{D(R)}{r^3} \right] P_\omega V \end{aligned}$$

where:

$$\begin{aligned} A(R) &= -\frac{3}{8R^3} + O\left(\frac{1}{R^4}\right), & B(R) &= \frac{9}{8R} + O\left(\frac{1}{R^2}\right), \\ C(R) &= -\frac{3}{4} + O\left(\frac{1}{R}\right), & D(R) &= \frac{1}{4} + O\left(\frac{1}{R}\right). \end{aligned}$$

The formula for u is obtained by replacing $A(R), B(R), C(R), D(R)$ by their limits when $R \rightarrow \infty$ in the formula defining u .

In the same spirit as on page 965 of [6], we have that, for arbitrary $x \in A(0, 1, R)$:

$$\begin{aligned} u_R(x) - u(x) &= \left[\frac{3}{2R^3}r^2 - r^2 O\left(\frac{1}{R^4}\right) - \frac{9}{4R} + O\left(\frac{1}{R}\right) + O\left(\frac{1}{R}\right)\frac{1}{r} + \frac{1}{r^3} O\left(\frac{1}{R}\right) \right] (\mathbb{I} - P_\omega)V \\ &+ \left[\frac{3}{4R^3}r^2 + r^2 O\left(\frac{1}{R^4}\right) - \frac{9}{4R} + O\left(\frac{1}{R}\right) + O\left(\frac{1}{R}\right)\frac{1}{r} + \frac{1}{r^3} O\left(\frac{1}{R}\right) \right] P_\omega V. \end{aligned}$$

This yields

$$\begin{aligned} & \int_{\partial B(0,1)} \nabla(u_R - u) n d\sigma \\ &= \left(\frac{3}{2R^3} + O\left(\frac{1}{R^4}\right) + O\left(\frac{1}{R}\right) \right) (4\pi^2 V - \int_{\partial B(0,1)} V \cdot x x d\sigma) \\ &+ \left(\frac{3}{2R^3} + O\left(\frac{1}{R^4}\right) + O\left(\frac{1}{R}\right) \right) \int_{\partial B(0,1)} V \cdot x x d\sigma, \end{aligned}$$

and consequently:

$$\left| \int_{\partial B(0,1)} \nabla(u_R - u) n d\sigma \right| \lesssim \frac{|V|}{R}.$$

By using [6, Section 6.2] we get a similary formula for the pressures:

$$\pi_R(x) - \pi(x) = \left(-20A(R)|x| + \frac{5A(R) + 3B(R)}{|x|^2} \right) \frac{x \cdot V}{|x|}, \quad \forall x \in A(0, 1, R).$$

This entails that:

$$\begin{aligned} \left| \int_{\partial B(0,1)} (\pi - \pi_R) n d\sigma \right| &\lesssim (25|A(R)| + 3|B(R)|) \left| \int_{\partial B(0,1)} V \cdot x d\sigma \right| \\ &\lesssim \frac{|V|}{R}. \end{aligned}$$

Finally, we get that:

$$|F_R^1 - F^1| \lesssim \frac{|V|}{R}.$$

We obtain the inequality for arbitrary r by remarking that, denoting $(\tilde{u}, \tilde{\pi})$ the solution to the Stokes problem on $\mathbb{R}^3 \setminus B(0, r)$ (resp. $(\tilde{u}_R, \tilde{\pi}_R)$ the solution to the Stokes problem on $A(0, r, R)$), we have:

$$\begin{aligned} (\tilde{u}(x), \tilde{\pi}(x)) &= \left(u\left(\frac{x}{r}\right), \frac{1}{r}\pi\left(\frac{x}{r}\right) \right), \quad \text{for all } x \in \mathbb{R}^3 \setminus B(0, r), \\ (\tilde{u}_R(x), \tilde{\pi}_R(x)) &= \left(u_{R/r}\left(\frac{x}{r}\right), \frac{1}{r}\pi_{R/r}\left(\frac{x}{r}\right) \right), \quad \text{for all } x \in A(0, r, R). \end{aligned}$$

Introducing this scaling in the formulas for F_R^r , we get that:

$$F_R^r - F^r = r(F_{R/r}^1 - F^1).$$

This entails finally that:

$$\begin{aligned} |F_R^r - F^r| &= r|F_{R/r}^1 - F^1| \\ &\lesssim r^2 \frac{|V|}{R}. \end{aligned}$$

□

We conclude this section by an error estimate for the velocity gradient:

Lemma 2.6. *There holds:*

$$\int_{A(0, R/2, R)} |\nabla u_R|^2 \lesssim \frac{r^2}{R} |V|^2.$$

Proof. We obtain the result for $r = 1$ by plugging the explicit formulas for u^R and the coefficients $A(R), B(R), C(R), D(R)$ in the previous proof and generalize it to arbitrary $r > 0$ by a scaling argument. The details are left to the reader. □

3. PROOFS OF THEOREM 1.1 AND 1.2

We proceed in this section with the proofs of our main theorems. In this section, we fix Ω, R_0, C_0 , and $p \in]1, 3/2[, q \in (3, \infty)$ as in the assumptions of our theorems. When using the symbol \lesssim , we allow the implicit constant to depend on these values R_0, C_0, p, q, Ω .

Let $N \geq N_0 := (4R_0/C_0)^{\frac{3}{2}}$. we recall that $E_\Omega[u^N]$ is the solution to the Stokes problem (1)-(2) on the perforated domain \mathcal{F}^N with boundary data V_1, \dots, V_N . With similar arguments to [11, Section 3] we have:

Proposition 3.1. *There exists a constant K depending only on R_0 and C_0 for which:*

$$\|E_\Omega[u^N]\| \leq KE_0, \quad \forall N \geq N_0.$$

We also introduce \bar{u} the solution to the Stokes-Brinkman problem (3)-(4) associated with the data j, ρ that may be computed from the particle distribution function to which the sequence of empirical measures describing the N -configurations converges.

The main idea is common to both proofs: we apply duality arguments reported in Lemma 2.3 or in Lemma 2.4 in order to estimate the L^p -norm of the vector-field $v^N := E_\Omega[u^N] - \bar{u}$. Hence, the core of the proof is the computation of

$$\left| \int_{\Omega} \nabla v^N : \nabla w \right|,$$

for an arbitrary divergence-free vector-field $w \in W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega)$.

In the two next parts, we prepare these computations by fixing a divergence-free vector-field $w \in W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega)$. We compute equivalent formulas for

$$\int_{\Omega} \nabla v^N : \nabla w,$$

and provide some bounds that are relevant for both proofs.

We remind the classical embedding that we use repeatedly below: since $p \in [1, \frac{3}{2}[$, there holds:

$$W^{2,p'}(\Omega) \hookrightarrow C^{0,1}(\bar{\Omega}).$$

3.1. Extraction of first order terms. Let $N \geq N_0$ and $w \in W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega)$ be divergence-free. We have:

$$\int_{\Omega} \nabla v^N : \nabla w = \int_{\Omega} \nabla E_\Omega[u^N] : \nabla w - \int_{\Omega} \nabla \bar{u} : \nabla w,$$

where

$$\begin{aligned} \int_{\Omega} \nabla \bar{u} : \nabla w &= \int_{\Omega} (j(x) - \rho(x)\bar{u}(x)) \cdot w(x) dx, \\ \int_{\Omega} \nabla E_\Omega[u^N] : \nabla w &= \int_{\mathcal{F}^N} \nabla u^N : \nabla w. \end{aligned}$$

In what follows we use the shortcuts:

$$(16) \quad A_j^N := A(x_j^N, C_0/4N^{1/3}, C_0/2N^{1/3}), \quad \bar{u}_j^N := \oint_{A_j^N} u^N.$$

and

$$\Omega_j^N = B\left(x_j^N, \frac{C_0}{2N^{1/3}}\right) \setminus B_j^N \quad \forall j, N.$$

Because of the definition (H1) of C_0 , the sets Ω_j^N are disjoint and cover a subset of Ω . These sets are also annuli but they play a special role to our proof hence the different name. Because $N \geq N_0$, the sets Ω_j^N are not empty so that their boundaries are made of two concentric spheres. The internal sphere is ∂B_j^N while we denote below $\partial_e \Omega_j^N$ the external sphere.

We first decompose the scalar product $\int_{\mathcal{F}^N} \nabla u^N : \nabla w$ into N integrals on the disjoint annuli Ω_j^N . To this end, given $j \in \{1, \dots, N\}$, we define $(\hat{w}_j^N, \hat{\pi}_j^N)$ the unique solution to the Stokes problem

$$(17) \quad \begin{cases} -\Delta \hat{w}_j^N + \nabla \hat{\pi}_j^N &= 0, \\ \operatorname{div} \hat{w}_j^N &= 0, \end{cases} \quad \text{on } \Omega_j^N,$$

completed with boundary conditions:

$$(18) \quad \begin{cases} \hat{w}_j^N(x) &= w(x), \quad \text{on } \partial B_j^N, \\ \hat{w}_j^N(x) &= 0, \quad \text{on } \partial_e \Omega_j^N. \end{cases}$$

We still denote \hat{w}_j^N the trivial extension of \hat{w}_j^N to \mathcal{F}^N and we set

$$w^N := \sum_{j=1}^N \hat{w}_j^N.$$

We remark then that w^N satisfies:

$$\begin{cases} w^N \in H^1(\mathcal{F}^N), \\ \operatorname{div} w^N = 0, \\ w^N = w, \end{cases} \quad \begin{array}{l} \text{on } \mathcal{F}^N, \\ \text{on } \partial \mathcal{F}^N. \end{array}$$

We have then:

$$\int_{\mathcal{F}^N} \nabla u^N : \nabla w = \int_{\mathcal{F}^N} \nabla u^N : \nabla (w - w^N) + \int_{\mathcal{F}^N} \nabla u^N : \nabla w^N.$$

Because u^N is the solution to the Stokes problem on \mathcal{F}^N and $w - w^N \in D_0(\mathcal{F}^N)$, the first term on the right-hand side vanishes:

$$\int_{\mathcal{F}^N} \nabla u^N : \nabla w = \sum_{j=1}^N \int_{\mathcal{F}^N} \nabla u^N : \nabla \hat{w}_j^N := \sum_{j=1}^N I_j^N.$$

Let denote now by (w_j^N, π_j^N) the unique solution to:

$$(19) \quad \begin{cases} -\Delta w_j^N + \nabla \pi_j^N &= 0, \\ \operatorname{div} w_j^N &= 0, \end{cases} \quad \text{on } \Omega_j^N,$$

completed with boundary conditions:

$$(20) \quad \begin{cases} w_j^N(x) &= w(x_j^N), & \text{on } \partial B_j^N, \\ w_j^N(x) &= 0, & \text{on } \partial_e \Omega_j^N. \end{cases}$$

For arbitrary $j = 1, \dots, N$, we have

$$I_j^N = \int_{\Omega_j^N} \nabla u^N : \nabla (\widehat{w}_j^N - w_j^N) + \int_{\Omega_j^N} \nabla u^N : \nabla w_j^N.$$

and we set

$$R1_j^N := \int_{\Omega_j^N} \nabla u^N : \nabla (\widehat{w}_j^N - w_j^N).$$

Because w_j^N is a solution to (19) and $u \in H^1(\Omega_j^N)$ is divergence-free we have also that:

$$I_j^N = \int_{\partial B_j^N} [(\nabla w_j^N - \pi_j^N \mathbb{I}) \cdot n] \cdot u^N d\sigma + \int_{\partial_e \Omega_j^N} [(\nabla w_j^N - \pi_j^N \mathbb{I}) \cdot n] \cdot u^N d\sigma + R1_j^N.$$

In the first integral, we note that $u^N = V_j^N$ on ∂B_j^N . We then introduce:

$$F_j^N = \int_{\partial B_j^N} (\nabla w_j^N - p \mathbb{I}) \cdot n d\sigma$$

to rewrite the first term:

$$\int_{\partial B_j^N} [(\nabla w_j^N - \pi_j^N \mathbb{I}) \cdot n] \cdot u^N d\sigma = F_j^N \cdot V_j^N.$$

As for the second term, we have (recall (16) for the definition of \bar{u}_j^N):

$$\int_{\partial_e \Omega_j^N} [(\nabla w_j^N - \pi_j^N \mathbb{I}) \cdot n] \cdot u^N d\sigma = \int_{\partial_e \Omega_j^N} [(\nabla w_j^N - \pi_j^N \mathbb{I}) \cdot n] \cdot \bar{u}_j^N d\sigma + R2_j^N,$$

where

$$R2_j^N = \int_{\partial_e \Omega_j^N} [(\nabla w_j^N - \pi_j^N \mathbb{I}) \cdot n] \cdot (u^N - \bar{u}_j^N) d\sigma.$$

At this point, we remark that the Stokes system is the divergence form of the conservation of the normal stresses. This yields that:

$$\int_{\partial B_j^N} (\nabla w_j^N - \pi_j^N \mathbb{I}) \cdot n d\sigma + \int_{\partial_e \Omega_j^N} (\nabla w_j^N - \pi_j^N \mathbb{I}) \cdot n d\sigma = 0.$$

Consequently, we obtain that:

$$\int_{\partial_e \Omega_j^N} [(\nabla w_j^N - \pi_j^N \mathbb{I}) \cdot n] \cdot u^N d\sigma = R2_j^N - F_j^N \cdot \bar{u}_j^N.$$

Eventually, plugging the identities above in $\int_{\mathcal{F}^N} \nabla u^N : \nabla w$ yields that:

$$(21) \quad \begin{aligned} \int_{\Omega} \nabla v^N : \nabla w &= \sum_{j=1}^N F_j^N \cdot V_j^N - \int_{\Omega} j(x) \cdot w(x) dx \\ &\quad - \left[\sum_{j=1}^N F_j^N \cdot \bar{u}_j^N - \int_{\Omega} \rho(x) \bar{u}(x) \cdot w(x) dx \right] \\ &\quad + R1^N + R2^N, \end{aligned}$$

where:

$$R1^N := \sum_{j=1}^N R1_j^N, \quad R2^N := \sum_{j=1}^N R2_j^N.$$

3.2. Estimates applied in both proofs. We state and prove here several propositions that are useful in the proof of both theorems.

Proposition 3.2. *There holds:*

$$\left| \sum_{k=1}^N F_k^N \cdot v_k^N - \int_{\Omega} j(x) \cdot w(x) dx \right| \lesssim \left(\frac{E_0}{N^{2/3}} + \|j - j^N\|_{(C^{0,1}(\bar{\Omega}))^*} \right) \|w\|_{2,p'}.$$

Proof. We define (W_j^N, Π_j^N) by:

$$(22) \quad (w_j^N(x), \pi_j^N(x)) = (W_j^N(N(x - x_j^N)), N\Pi_j^N(N(x - x_j^N))), \quad \forall x \in \Omega_j^N.$$

We note that, substituting in the integral yields:

$$\int_{\partial B_j^N} (\nabla w_j^N - \pi_j^N \mathbb{I}) \cdot n d\sigma = \frac{1}{N} \int_{\partial B(0, r_j^N)} (\nabla W_j^N - \Pi_j^N \mathbb{I}) \cdot n d\sigma,$$

and:

$$\begin{aligned} F_j^N &= \frac{1}{N} \int_{\partial B(0, r_j^N)} (\nabla W_j^N - \Pi_j^N \mathbb{I}) \cdot n d\sigma \\ &= \frac{1}{N} \left(\int_{\partial B(0, r_j^N)} (\nabla W_j^N - \Pi_j^N \mathbb{I}) \cdot n d\sigma - 6\pi r_j^N w(x_j^N) \right) + \frac{6\pi}{N} r_j^N w(x_j^N). \end{aligned}$$

We remark then that (W_j^N, Π_j^N) is solution to:

$$(23) \quad \begin{cases} -\Delta W_j^N + \nabla \Pi_j^N &= 0, \\ \operatorname{div} W_j^N &= 0, \end{cases} \quad \text{on } B(0, C_0/N^{2/3}) \setminus B(0, r_j^N),$$

completed with boundary conditions:

$$(24) \quad W_j^N(x) = w(x_j^N), \text{ on } \partial B(0, r_j^N), \quad W_j^N(x) = 0, \text{ on } \partial B(0, C_0/N^{2/3}),$$

so that Lemma 2.5 applies. Assumptions (H2) and (H3) then entail that:

$$\begin{aligned} \left| \sum_{k=1}^N F_k^N \cdot v_k^N - \int_{\Omega} j \cdot w \right| &= \left| \frac{1}{N} \left(\sum_{j=1}^N \int_{\partial B(0, r_j^N)} (\nabla W_j^N - P_j^N \mathbb{I}) \cdot n d\sigma - 6\pi r_j^N w(x_j^N) \right) \cdot v_k^N \right. \\ &\quad \left. + \frac{6\pi}{N} \sum_{k=1}^N r_k^N [w(x_k^N)] \cdot v_k^N - \int_{\Omega} j \cdot w \right| \\ &\lesssim \frac{1}{N} \sum_{k=1}^N \frac{|r_k^N|^2}{N^{2/3}} |w(x_k^N)| |v_k^N| + |\langle j^N - j, w \rangle| \\ &\lesssim \frac{E_0}{N^{2/3}} \|w\|_{\infty} + \|j^N - j\|_{(C^{0,1}(\bar{\Omega}))^*} \|w\|_{C^{0,1}(\bar{\Omega})}. \end{aligned}$$

We conclude the proof by applying the embedding $W^{2,p'}(\Omega) \subset C^{0,1}(\bar{\Omega})$. \square

Remark 3.1. A more general estimate can be proved when $p \in]3/2, 3[$. Indeed, in this case we have the Sobolev embedding $W^{2,p'}(\Omega) \hookrightarrow C^{0,\min(1,\alpha_p)}(\bar{\Omega})$ with $\alpha_p := 2 - \frac{3}{p'} = -1 + \frac{3}{p} \in (0, 1)$. Hence, in the last list of inequality, we may bound:

$$|\langle j^N - j, w \rangle| \leq \|j^N - j\|_{(C^{0,\alpha_p}(\bar{\Omega}))^*} \|w\|_{2,p'}.$$

We complete the joint part of our main proofs by showing that both $R1^N$ and $R2^N$ vanish when $N \rightarrow \infty$. First, we have the following proposition:

Proposition 3.3. *There holds*

$$|R1^N| \lesssim \frac{E_0 \|w\|_{2,p'}}{N}.$$

Proof. We remind that:

$$R1^N = \sum_{j=1}^N \int_{\Omega} \nabla E_{\Omega}(u^N) : \nabla (\hat{w}_j^N - w_j^N).$$

We set \tilde{w}_j^N the difference $\hat{w}_j^N - w_j^N$, hence:

$$\begin{aligned} |R1^N| &\leq \|\nabla E_{\Omega}(u^N)\|_{L^2(\Omega)} \left(\sum_{j=1}^N \|\nabla \tilde{w}_j^N\|_{L^2(\Omega_j^N)}^2 \right)^{1/2} \\ (25) \quad &\lesssim E_0 \left(\sum_{j=1}^N \|\nabla \tilde{w}_j^N\|_{L^2(\Omega_j^N)}^2 \right)^{1/2}, \end{aligned}$$

because of the bound on $E_{\Omega}[u^N]$ that we obtained in Proposition 3.1.

At this point, we remark that, for $j \in \{1, \dots, N\}$ the \tilde{w}_j^N can be associated with a pressure $\tilde{\pi}_j^N$ (namely $\hat{\pi}_j^N - \pi_j^N$) to get the unique solution to the Stokes problem:

$$(26) \quad \begin{cases} -\Delta \tilde{w}_j^N + \nabla \tilde{\pi}_j^N &= 0, \\ \operatorname{div} \tilde{w}_j^N &= 0, \end{cases} \quad \text{on } \Omega_j^N,$$

completed with boundary conditions:

$$(27) \quad \begin{cases} \tilde{w}_j^N(x) &= w(x) - w(x_j), & \text{on } \partial B_j^N, \\ \tilde{w}_j^N &= 0 & \text{on } \partial_e \Omega_j^N. \end{cases}$$

The aim is to bound the $H_0^1(\Omega_j^N)$ -norm of \tilde{w}_j^N by constructing a lifting of boundary conditions (27) and using the variational characterization of \tilde{w}_j^N solution to (26)-(27).

Let χ be a truncation function equal to 1 on $B(0, R_0)$ and vanishing outside $B(0, 2R_0)$. We set $\chi^N := \chi(N(x - x_j^N))$ and we denote $v = v_1 + v_2$ where:

$$\begin{aligned} v_1(x) &= \chi^N(x)(w(x) - w(x_j^N)), \quad \forall x \in \Omega_j^N, \\ v_2 &= \mathfrak{B}_{x_j, R_0/N, 2R_0/N}[-\operatorname{div}(v_1)], \end{aligned}$$

with \mathfrak{B} the bogovskii operator (see [7, Section III.3]). Because $\operatorname{div}(v_1) = \nabla \chi^N \cdot (w - w(x_j^N))$ has mean 0 on Ω_j^N , the vector-field v_2 is well-defined. We may then apply [11, Appendix A, Lemma 15] to get that:

$$\begin{aligned} \int_{\Omega_j^N} |\nabla v|^2 &\lesssim \int_{A(x_j^N, R_0/N, 2R_0/N)} |\nabla w(x)|^2 \\ &+ N^2 \int_{A(x_j^N, R_0/N, 2R_0/N)} |w(x) - w(x_j^N)|^2 \sup_{x \in B(0, 2R_0)} |\nabla \chi(x)|^2 \\ &\lesssim \frac{1}{N^3} \|w\|_{W^{2,p'}(\Omega)}^2. \end{aligned}$$

We applied here again the embedding $W^{2,p'}(\Omega) \hookrightarrow \mathcal{C}^{0,1}(\bar{\Omega})$ for $p' > 3$. Finally, we have

$$|R1^N| \lesssim \frac{E_0}{N} \|w\|_{W^{2,p'}(\Omega)}.$$

This ends the proof of our estimate. \square

Remark 3.2. *As in Remark 3.1, a more general result can be obtained for all $p \in]3/2, 3[$. In this case, we have that $W^{2,p'}(\Omega) \hookrightarrow \mathcal{C}^{0,\alpha_p}(\bar{\Omega})$, which provides a more general bound for the error term $R1^N$ of the form $\frac{1}{N^{\frac{1}{\alpha_p}}}$.*

In order to compute the second error term, we need the following lemma. We recall that the annuli A_j^N are defined in (16). We keep the convention that $\partial_e A_j^N$ stands for the external sphere bounding A_j^N .

Lemma 3.4. *For $j = 1, \dots, N$, let $v_j^N \in H^1(A_j^N)$ satisfy:*

- $\operatorname{div} v_j^N = 0$ on A_j^N ;

- the flux of v_j^N through the exterior boundary of A_j^N vanishes:

$$\int_{\partial_e A_j^N} v_j^N \cdot n d\sigma = 0;$$

- the mean of v_j^N on A_j^N vanishes.

Then, there holds:

$$\left| \sum_{j=1}^N \int_{\partial_e \Omega_j^N} (\nabla w_j^N - \pi_j^N \mathbb{I}) n \cdot v_j^N d\sigma \right| \lesssim \left(\sum_{j=1}^N \|\nabla v_j^N\|_{L^2(A_j^N)}^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^N \|\nabla w_j^N\|_{L^2(A_j^N)}^2 \right)^{\frac{1}{2}}.$$

Proof. We begin by introducing a suitable lifting of $v_j^N|_{\partial_e A_j^N}$.

Namely, we introduce a truncation function χ such that χ vanishes on $B(0, C_0/4)$ and is equal to 1 outside $B(0, C_0/3)$. For $j \in \{1, \dots, N\}$, we denote $\chi_j^N = \chi(N^{1/3}(x - x_j^N))$ and we set:

$$\tilde{v}_j = \tilde{v}_{j,1} + \tilde{v}_{j,2},$$

where:

$$\tilde{v}_{j,1} = \chi_j^N v_j^N, \quad \tilde{v}_{j,2} = \mathfrak{B}_{x_j^N, C_0/4N^{1/3}, C_0/2N^{1/3}}[-\operatorname{div} \tilde{v}_{j,1}].$$

As, by assumption, we have that v_j^N has flux zero on $\partial_e A_j^N$ we obtain that $\operatorname{div} \tilde{v}_{j,1}$ has mean zero on A_j^N and $\tilde{v}_{j,2}$ is well-defined. For convenience, we also set:

$$w^N = \sum_{j=1}^N \mathbf{1}_{\Omega_j^N} w_j^N, \quad D\tilde{v} = \sum_{j=1}^N \mathbf{1}_{\Omega_j^N} \nabla \tilde{v}_j.$$

At this point, we note that:

- on $\partial B_j^N \subset B(x_j^N, C_0/4N^{1/3})$ we have $\chi_j^N = 0$ so that $\tilde{v}_{j,1} = 0$. As $\tilde{v}_{j,2} = 0$ by construction, we get $\tilde{v}_j = 0$,
- on $\partial_e A_j^N = \partial_e \Omega_j^N$, we have $\chi_j^N = 1$ so that $\tilde{v}_{j,1} = v_j^N$. As, by construction, $\tilde{v}_{j,2} = 0$, we get $v_j^N = \tilde{v}_j$.

These remarks entail that:

$$\begin{aligned} \sum_{j=1}^N \int_{\partial_e A_j^N} (\nabla w_j^N - \pi_j^N \mathbb{I}) n \cdot v_j^N d\sigma &= \sum_{j=1}^N \int_{\partial \Omega_j^N} (\nabla w_j^N - \pi_j^N \mathbb{I}) n \cdot \tilde{v}_j d\sigma \\ &= \sum_{j=1}^N \int_{\Omega_j^N} \nabla w_j^N : \nabla \tilde{v}_j, \\ &= \int_{\Omega} \mathbf{1}_{\operatorname{Supp}(D\tilde{v})} \nabla w^N : D\tilde{v}. \end{aligned}$$

Consequently, we have:

$$\left| \sum_{j=1}^N \int_{\partial_e \Omega_j^N} (\nabla w_j^N - \pi_j^N \mathbb{I}) n \cdot v_j^N d\sigma \right| \leq \left(\int_{\Omega} \mathbf{1}_{\operatorname{Supp}(D\tilde{v})} |\nabla w^N|^2 \right)^{\frac{1}{2}} \left(\int_{\Omega} |D\tilde{v}|^2 \right)^{\frac{1}{2}}.$$

Due to the fact that the supports of the Ω_j^N are disjoint and cover the support of $D\tilde{v}$, we have:

$$\int_{\Omega} \mathbf{1}_{\text{Supp}(D\tilde{v})} |\nabla w^N|^2 = \sum_{j=1}^N \int_{\Omega_j^N} \mathbf{1}_{\text{Supp}(D\tilde{v})} |\nabla w_j^N|^2$$

where $\text{Supp}(D\tilde{v}) \cap \Omega_j^N = A_j^N$ so that

$$\int_{\Omega} \mathbf{1}_{\text{Supp}(D\tilde{v})} |\nabla w^N|^2 = \sum_{j=1}^N \int_{A_j^N} |\nabla w_j^N|^2.$$

With a similar decomposition, we obtain also:

$$\begin{aligned} \int_{\Omega} |D\tilde{v}|^2 &= \sum_{j=1}^N \int_{A_j^N} |\nabla \tilde{v}_j|^2, \\ &\leq 2 \sum_{j=1}^N \int_{A_j^N} |\nabla \tilde{v}_{j,1}|^2 + |\nabla \tilde{v}_{j,2}|^2. \end{aligned}$$

As in the proof of the previous proposition, we compute the terms $\nabla \tilde{v}_{j,1}$, $\nabla \tilde{v}_{j,2}$ and use estimates on Bogovskii operator (see [11, Appendix A, lemma 15]) to get that there exists a positive constant K_{χ} such that:

$$\int_{A_j^N} |\nabla \tilde{v}_{j,1}|^2 + |\nabla \tilde{v}_{j,2}|^2 \leq K_{\chi} \left(\int_{A_j^N} N^{2/3} |v_j^N|^2 + \int_{A_j^N} |\nabla v_j^N|^2 \right).$$

Finally, we apply the Poincaré-Wirtinger inequality in the case of annuli (see [11, Appendix A, Lemma 13]): there exists a constant $C > 0$ independent of N for which:

$$\int_{A_j^N} |v_j^N|^2 \leq C \left(\frac{C_0}{2N^{1/3}} \right)^2 \int_{A_j^N} |\nabla v_j^N|^2.$$

Finally we get that

$$\int_{\Omega} |D\tilde{v}|^2 \lesssim \sum_{j=1}^N \int_{A_j^N} |\nabla v_j^N|^2.$$

□

We may now state the result on the control of the second error term R_2^N :

Proposition 3.5. *There holds*

$$|R_2^N| \lesssim \frac{E_0 \|w\|_{2,p'}}{N^{1/3}}.$$

Proof. The main idea to compute R_2^N is to apply the previous lemma to

$$v_j^N = \mathbf{1}_{A_j^N} \left[u^N - \oint_{A(x_j^N, C_0/4N^{1/4}, C_0/2N^{1/3})} u^N \right].$$

This entails that

$$(28) \quad |R2^N| \leq K \|\nabla u^N\|_{L^2(\mathcal{F}^N)} \left(\sum_{i=1}^N \|\nabla w_j^N\|_{L^2(A_j^N)}^2 \right)^{\frac{1}{2}}.$$

At this point, we recall the definition of W_j^N (see (22)) and use the change of variable $y = N(x - x_j^N)$:

$$\begin{aligned} \|\nabla w_j^N\|_{L^2(A_j^N)}^2 &= \int_{A_j^N} N^2 |\nabla W_j^N(N(x - x_j^N))|^2 dx \\ &= \frac{1}{N} \int_{A(0, \frac{C_0 N^{2/3}}{4}, \frac{C_0 N^{2/3}}{2})} |\nabla W_j^N(y)|^2 dy \\ &= \frac{1}{N} \|\nabla W_j^N\|_{L^2(A(0, \frac{C_0 N^{2/3}}{4}, \frac{C_0 N^{2/3}}{2}))}^2. \end{aligned}$$

We may then apply Lemma 2.6 to get that:

$$\|\nabla W_j^N\|_{L^2(A(0, \frac{C_0 N^{2/3}}{4}, \frac{C_0 N^{2/3}}{2}))}^2 \lesssim |r_j^N|^2 \frac{|w(x_j^N)|^2}{C_0 N^{2/3}}.$$

Plugging these identities into (28), applying the fact that $E_\Omega(u^N)$ is bounded for the $D_0(\Omega)$ -norm and assumption (H2), we obtain:

$$|R2^N| \lesssim E_0 \left(\frac{1}{N} \sum_{j=1}^N |r_j^N|^2 \frac{|w(x_j^N)|^2}{C_0 N^{2/3}} \right)^{1/2} \lesssim \frac{E_0}{N^{1/3}} \|w\|_\infty.$$

□

3.3. Proof of Theorem 1.1. We now turn to the proof of the theorem including a smallness assumption on the size of the holes. For this proof, we first complement the computations in the previous section by estimating the term on the second line of (21):

Proposition 3.6. *Under the further assumption that $j \in L^q(\Omega)$ for some $q > 3$, there exists $K_{p,\Omega}$ depending only on p and Ω such that:*

$$\begin{aligned} \left| \sum_{k=1}^N F_k^N \cdot \bar{u}_k^N - \int_\Omega \rho(x) w(x) \cdot \bar{u}(x) \right| - K_{p,\Omega} \frac{R_0}{C_0^3} \|w\|_{2,p'} \|u^N - \bar{u}\|_p \\ \lesssim \left[\frac{E_0}{N^{2/3}} + \left(\|\rho^N - \rho\|_{(C^{0,1}(\bar{\Omega}))^*} + \frac{1}{N^{1/3}} \right) \|\bar{u}\|_{2,3} \right] \|w\|_{2,p'} \end{aligned}$$

Proof. We may write:

$$\sum_{k=1}^N F_k^N \cdot \bar{u}_k^N = \sum_{k=1}^N \left(F_k^N - \frac{6\pi}{N} r_k^N w(x_k^N) \right) \cdot \bar{u}_k^N + \frac{6\pi}{N} r_k^N w(x_k^N) \cdot \bar{u}_k^N$$

We remind that given $k \in \{1, \dots, N\}$:

$$|A_k^N| = |B(x_k^N, C_0/2N^{1/3})| - |B(x_k^N, C_0/4N^{1/3})| = \frac{4}{3}\pi \left(\frac{C_0^3}{8N} - \frac{C_0^3}{64N} \right) = \frac{7C_0^3\pi}{48N}.$$

According to the same computations as in the proof of Proposition 3.2:

$$\begin{aligned} \left| \sum_{k=1}^N \left(F_k^N - \frac{6\pi}{N} r_k^N w(x_k^N) \right) \cdot \bar{u}_k^N \right| &\lesssim \frac{1}{N^{2/3}} \sum_{k=1}^N \frac{1}{N} |\bar{u}_k^N| \|w\|_\infty \\ &\lesssim \frac{1}{N^{2/3}} \|w\|_\infty \frac{1}{N} \sum_{k=1}^N \frac{1}{|A_k^N|} \int_{A_k^N} |u^N| \\ &\lesssim \frac{E_0}{N^{2/3}} \|w\|_\infty. \end{aligned}$$

In order to compute the remaining term we introduce the linear mapping:

$$\Pi_N : \begin{cases} \mathcal{C}_c^\infty(\bar{\Omega}) & \longrightarrow \mathbb{R} \\ \phi & \longmapsto \langle \Pi_N, \phi \rangle := \frac{6\pi}{N} \sum_{k=1}^N r_k^N w(x_k^N) \cdot \oint_{A_k^N} \phi. \end{cases}$$

We also set:

$$\langle \Pi, \phi \rangle := \int_{\Omega} \rho(x) w(x) \cdot \phi(x) dx,$$

to rewrite the term:

$$\begin{aligned} \frac{1}{N} \sum_{k=1}^N 6\pi r_k^N w(x_k^N) \cdot \bar{u}_k^N - \int_{\Omega} \rho(x) w(x) \cdot \bar{u}(x) dx &= \langle \Pi_N, u^N \rangle - \langle \Pi, \bar{u} \rangle \\ &= \langle \Pi_N, u^N - \bar{u} \rangle + \langle \Pi_N - \Pi, \bar{u} \rangle. \end{aligned}$$

By straightforward computations, we show that $(\Pi_N)_N$ is a bounded family of linear mappings on $L^p(\Omega)$. Indeed, recalling the definition of R_0 in (H2) and the above computation of $|A_k^N|$, we obtain:

$$\begin{aligned} |\langle \Pi_N, \phi \rangle| &\leq \|w\|_\infty \frac{6\pi}{N|A_1^N|} \max_k r_k^N \int_{\bigsqcup_k A_k^N} |\phi| \\ &\leq K_{p,\Omega} \frac{R_0}{C_0^3} \|w\|_\infty \|\phi\|_p, \end{aligned}$$

with $K_{p,\Omega}$ depending only on Ω and p . Hence, applying the embedding $W^{2,p'}(\Omega) \subset L^\infty(\Omega)$ (with a constant depending only on p, Ω) we obtain with a possibly different constant $K_{p,\Omega}$, keeping the same dependencies:

$$|\langle \Pi_N, u^N - \bar{u} \rangle| \leq K_{p,\Omega} \frac{R_0}{C_0^3} \|w\|_{2,p'} \|u^N - \bar{u}\|_p.$$

To compute the last term we use the regularity of \bar{u} solution to the Brinkman problem. Indeed, if $j \in L^q$ for some $q > 3$ then Theorem 2.2 shows that $\bar{u} \in W^{2,q}(\Omega) \hookrightarrow \mathcal{C}^{0,1}(\bar{\Omega})$, thus, there holds:

$$\begin{aligned}
|\langle \Pi_N - \Pi, \bar{u} \rangle| &= \left| \frac{6\pi}{N} \sum_{k=1}^N r_k^N w(x_k^N) \cdot \bar{u}(x_k^N) - \int_{\Omega} \rho(x) w(x) \cdot \bar{u}(x) dx \right. \\
&\quad \left. + \frac{6\pi}{N} \sum_{k=1}^N r_k^N w(x_k^N) \cdot \oint_{A_k^N} (\bar{u} - \bar{u}(x_k^N)) \right| \\
&\lesssim |\langle \rho^N - \rho, w \cdot \bar{u} \rangle| \\
&\quad + \frac{6\pi}{N} \sum_{k=1}^N r_k^N |w(x_k^N)| \frac{C_0}{2N^{1/3}} \|\bar{u}\|_{\mathcal{C}^{0,1}(\bar{\Omega})} \\
&\lesssim \|\rho^N - \rho\|_{(\mathcal{C}^{0,1}(\bar{\Omega}))^*} \|w\|_{\mathcal{C}^{0,1}(\bar{\Omega})} \|\bar{u}\|_{\mathcal{C}^{0,1}(\bar{\Omega})} \\
&\quad + \frac{1}{N^{1/3}} \|w\|_{\infty} \|\bar{u}\|_{\mathcal{C}^{0,1}(\bar{\Omega})}.
\end{aligned}$$

□

Remark 3.3. When $j \in L^q(\Omega)$ with $q \in]3/2, 3[$, a similar estimate holds involving the distance between ρ and ρ^N in the dual of $\mathcal{C}^{0,\alpha_q}(\bar{\Omega})$. This restriction is due to the embedding $\bar{u} \in W^{2,2}(\Omega) \hookrightarrow \mathcal{C}^{0,\alpha_q}(\bar{\Omega})$. The case $q \in]3/2, 3[$ involves also a remainder term that converges to zero like $\frac{1}{N^{\alpha_q/3}}$.

To complete the proof of Theorem 1.1, we remind that we introduced an exponent $p \in]1, 3/2[$, and an arbitrary divergence-free test-function $w \in W_0^{1,p'}(\Omega) \cap W^{2,p'}(\Omega)$; inspired by Lemma 2.3, we computed (21) which we recall here:

$$\begin{aligned}
(29) \quad \int_{\Omega} \nabla(E_{\Omega}[u^N] - \bar{u}) : \nabla w &= \left(\sum_{j=1}^N F_j^N \cdot v_j^N - \int_{\Omega} j(x) \cdot w(x) dx \right) \\
&\quad + \left(\int_{\Omega} \rho(x) \bar{u}(x) \cdot w(x) dx - \sum_{j=1}^N F_j^N \cdot \bar{u}_j^N \right) + R1^N + R2^N.
\end{aligned}$$

At this point, we apply now propositions 3.2, 3.3, 3.5 and 3.6. This entails that there exists a constant K depending only on p, Ω, R_0, C_0 for which

$$\left| \int_{\Omega} \nabla(E_{\Omega}[u^N] - \bar{u}) : \nabla w \right| \leq \|w\|_{2,p'} \left(K \left[\|j^N - j\|_{(C^{0,1}(\bar{\Omega}))^*} + \frac{E_0 + \|\bar{u}\|_{2,q}}{N^{1/3}} + \|\rho^N - \rho\|_{(C^{0,1}(\Omega))^*} \right] + K_{p,\Omega} \frac{R_0}{C_0^3} \|u^N - \bar{u}\|_p \right).$$

Consequently, applying Lemma 2.3 and regularity theory for Stokes-Brinkman problem, we obtain a constant $K_{p,\Omega}$ which may differ from the previous ones, but still depending only on p and Ω , such that:

$$\left(1 - K_{p,\Omega} \frac{R_0}{C_0^3} \right) \|u^N - \bar{u}\|_p \lesssim \|j^N - j\|_{(C^{0,1}(\bar{\Omega}))^*} + \|\rho^N - \rho\|_{(C^{0,1}(\Omega))^*} + \frac{E_0 + \|j\|_q}{N^{1/3}}.$$

This yields the expected result assuming that R_0/C_0^3 is sufficiently small.

3.4. Proof of Theorem 1.2. We proceed with the proof of our second main result. We do not consider in this case any particular restriction on the ratio R_0/C_0^3 . We want to apply now Lemma 2.4. So, we remind that for a fixed divergence-free test-function $w \in W_0^{1,p'}(\Omega) \cap W^{2,p'}(\Omega)$, by using again formula (21), we get:

$$(30) \quad \int_{\Omega} \nabla[E_{\Omega}[u^N] - \bar{u}] : \nabla w + \int_{\Omega} \rho[E_{\Omega}[u^N] - \bar{u}] \cdot w = \left(\sum_{j=1}^N F_j^N \cdot v_j^N - \int_{\Omega} j \cdot w \right) + \left(\int_{\Omega} \rho E_{\Omega}[u^N] \cdot w - \sum_{j=1}^N F_j^N \cdot \bar{u}_j^N \right) + R1^N + R2^N.$$

In order to treat the new term

$$\int_{\Omega} \rho E_{\Omega}[u^N] \cdot w - \sum_{j=1}^N F_j^N \cdot \bar{u}_j^N,$$

we apply the following proposition:

Proposition 3.7. *There holds:*

$$\left| \int_{\Omega} \rho E_{\Omega}[u^N] \cdot w - \sum_{j=1}^N F_j^N \cdot \bar{u}_j^N \right| \lesssim \left(\|\rho - \rho^N\|_{(C^{0,1}(\bar{\Omega}))^*} + \frac{1}{N^{1/3}} \right)^{1/3} E_0 \|w\|_{W^{2,p'}(\Omega)}.$$

Proof. Using the same notations Π_N and Π as in the previous proof, we write:

$$\int_{\Omega} \rho E_{\Omega}[u^N] \cdot w - \sum_{j=1}^N F_j^N \cdot \bar{u}_j^N = \sum_{k=1}^N \left(\frac{6\pi}{N} r_k^N w(x_k^N) - F_k^N \right) \cdot \bar{u}_k^N + \langle \Pi - \Pi_N, E_{\Omega}[u^N] \rangle,$$

where the first quantity on the right-hand side is treated as in Proposition 3.2:

$$\left| \sum_{k=1}^N (F_k^N - \frac{6\pi}{N} r_k^N w(x_k^N)) \cdot \bar{u}_k^N \right| \lesssim \frac{E_0}{N^{2/3}} \|w\|_\infty.$$

To compute the second term $|\langle \Pi_N - \Pi, u^N \rangle|$, we remark that for arbitrary smooth test function ϕ there holds:

$$\begin{aligned} \langle \Pi_N - \Pi, \phi \rangle &= \frac{6\pi}{N} \sum_{j=1}^N r_j^N w(x_j^N) \cdot \oint_{A_j^N} \phi(x) dx - \langle \rho, \phi \cdot w \rangle \\ &= \langle \rho^N - \rho, \phi \cdot w \rangle + \frac{6\pi}{N} \sum_{j=1}^N r_j^N w(x_j^N) \oint_{A_j^N} (\phi(x) - \phi(x_j^N)) dx, \end{aligned}$$

and consequently,

$$\begin{aligned} |\langle \Pi_N - \Pi, \phi \rangle| &\lesssim |\langle \rho^N - \rho, \phi \cdot w \rangle| + \frac{1}{N^{1/3}} \|w\|_\infty \|\phi\|_{C^{0,1}(\bar{\Omega})} \\ &\lesssim \left(\|\rho - \rho^N\|_{(C^{0,1}(\bar{\Omega}))^*} + \frac{1}{N^{1/3}} \right) \|\phi\|_{C^{0,1}(\bar{\Omega})} \|w\|_{C^{0,1}(\bar{\Omega})}. \end{aligned}$$

On the other hand for all $\phi \in L^2(\Omega)$, we have that:

$$\begin{aligned} |\langle \Pi_N - \Pi, \phi \rangle| &= \left| \frac{6\pi}{N} \int_{\Omega} \sum_{j=1}^N \frac{1_{A_j^N}(x)}{|A_j^N|} w(x_j^N) \cdot \phi(x) dx - \int_{\Omega} \rho \phi \cdot w \right| \\ &\lesssim \|w\|_\infty \sum_{j=1}^N \int_{A_j^N} |\phi| + \|w\|_\infty \|\rho\|_2 \|\phi\|_2 \\ &\lesssim \|w\|_\infty \|\phi\|_2. \end{aligned}$$

We now propose to interpolate the results above as we want to apply the previous inequalities with $\phi = E_\Omega[u^N] \in H_0^1(\Omega)$. So, let χ be a mollifier having support in $B(0, 1)$. We construct then the approximation of unity

$$\chi_\delta(\cdot) = \frac{1}{\delta^3} \chi\left(\frac{\cdot}{\delta}\right), \quad \forall \delta > 0.$$

Thanks to the previous computations, we have that:

$$\begin{aligned} |\langle \Pi_N - \Pi, E_\Omega[u^N] \rangle| &\leq |\langle \Pi_N - \Pi, E_\Omega[u^N] * \chi_\delta \rangle| + |\langle \Pi_N - \Pi, E_\Omega[u^N] - E_\Omega[u^N] * \chi_\delta \rangle| \\ &\lesssim \left(\|\rho - \rho^N\|_{(C^{0,1}(\bar{\Omega}))^*} + \frac{1}{N^{1/3}} \right) \|E_\Omega[u^N] * \chi_\delta\|_{C^{0,1}(\bar{\Omega})} \|w\|_{C^{0,1}(\bar{\Omega})} \\ &\quad + \|w\|_\infty \|E_\Omega[u^N] - E_\Omega[u^N] * \chi_\delta\|_2. \end{aligned}$$

At this point we remark that $E_\Omega[u^N] * \chi_\delta \in H^3(\Omega) \hookrightarrow \mathcal{C}^{0,1}(\bar{\Omega})$ with continuous embedding. Furthermore, straightforward computations show that:

$$\begin{aligned} \|E_\Omega[u^N] - E_\Omega[u^N] * \chi_\delta\|_{L^2(\mathbb{R}^3)} &\lesssim \delta \|u\|_{H_0^1(\Omega)}, \\ \|E_\Omega[u^N] * \chi_\delta\|_{H^3(\mathbb{R}^3)} &\lesssim \frac{1}{\delta^2} \|u\|_{H_0^1(\Omega)}. \end{aligned}$$

Plugging these estimates in the previous inequality yields that:

$$\begin{aligned} &|\langle \Pi_N - \Pi, E_\Omega[u^N] \rangle| \\ &\leq \left(\|\rho - \rho^N\|_{(\mathcal{C}^{0,1}(\bar{\Omega}))^*} + \frac{1}{N^{1/3}} \right) \frac{1}{\delta^2} \|\nabla E_\Omega[u^N]\|_2 \|w\|_{\mathcal{C}^{0,1}} + \delta \|w\|_{L^\infty} \|\nabla E_\Omega[u^N]\|_{L^2(\Omega)}. \end{aligned}$$

We may then set $\delta = \left(\|\rho - \rho^N\|_{(\mathcal{C}^{0,1}(\bar{\Omega}))^*} + \frac{1}{N^{1/3}} \right)^{1/3}$ and again apply that $W^{2,p'}(\Omega) \subset \mathcal{C}^{0,1}(\bar{\Omega})$ with the uniform control on $\|\nabla E_\Omega[u^N]\|_2$ to get that

$$|\langle \Pi_N - \Pi, E_\Omega[u^N] \rangle| \lesssim \left(\|\rho - \rho^N\|_{(\mathcal{C}^{0,1}(\bar{\Omega}))^*} + \frac{1}{N^{1/3}} \right)^{1/3} E_0 \|w\|_{W^{2,p'}(\Omega)}.$$

□

Similarly as in the proof of the previous theorem, we complete the proof of Theorem 1.2 by applying propositions 3.2, 3.3, 3.5 and 3.7 to control the right-hand side of (30) and referring to Lemma 2.4 to conclude.

4. FINAL REMARKS

In the main theorems of this paper, we measure the distance $E_\Omega[u^N] - \bar{u}$ in L^p -spaces with respect to the distances between (ρ^N, j^N) and (ρ, j) in the bounded-Lipschitz norms. With the same method, we may prove similar estimates when considering some Zolotarev-like distances of the data:

$$\|\rho^N - \rho\|_{\mathcal{C}^{0,\alpha}(\bar{\Omega})} + \|j^N - j\|_{\mathcal{C}^{0,\alpha}(\bar{\Omega})}, \quad \alpha \in (0, 1).$$

Precisely, reproducing the computations of the paper and introducing the remarks 3.1, 3.2 and 3.3, we may prove:

Theorem 4.1. *Let $\alpha \in (0, 1)$ and $(p, q) \in (1, 3/(1 + \alpha)) \times (3/(2 - \alpha), \infty)$. Assume that $j \in L^q(\Omega)$ and R_0/C_0^3 is sufficiently small, there exists a constant $K > 0$ depending only on R_0, C_0, p, q, Ω for which:*

$$\|E_\Omega[u^N] - \bar{u}\|_{L^p(\Omega)} \leq K \left[\|j^N - j\|_{(\mathcal{C}^{0,\alpha}(\bar{\Omega}))^*} + \|\rho^N - \rho\|_{(\mathcal{C}^{0,\alpha}(\bar{\Omega}))^*} + \frac{E_0}{N^{\min(1/3, \alpha)}} + \frac{\|j\|_{L^q(\Omega)}}{N^{\alpha/3}} \right],$$

for $N \geq (4R_0/C_0)^{3/2}$.

Theorem 4.2. *Let $\alpha \in (0, 1)$ and $p \in (1, 3/(1 + \alpha))$. There exists a constant $K > 0$ depending only on $R_0, C_0, p, \|\rho\|_{L^\infty(\Omega)}, \Omega$ for which:*

$$\|E_\Omega[u^N] - \bar{u}\|_{L^p(\Omega)} \leq K \left[\|j - j^N\|_{(\mathcal{C}^{0,\alpha}(\bar{\Omega}))^*} + \left(\|\rho - \rho^N\|_{(\mathcal{C}^{0,\alpha}(\bar{\Omega}))^*} + \frac{1}{N^{1/3}} \right)^{1/3} E_0 \right],$$

for $N \geq (4R_0/C_0)^{3/2}$.

APPENDIX A. FLUID/SOLID INTERACTION

In this part, we assume that $\Omega = \mathbb{R}^3 \setminus B(0, r)$ where $r > 0$. Let $V \in \mathbb{R}^3$ be fixed in what follows. We consider the unique pair (u, π) solution to the Stokes problem:

$$(31) \quad \begin{cases} -\Delta u + \nabla \pi &= 0, \\ \operatorname{div} u &= 0, \end{cases} \quad \text{on } \Omega,$$

completed with boundary conditions:

$$(32) \quad u(x) = V, \text{ on } \partial B(0, r), \quad \lim_{|x| \rightarrow \infty} u(x) = 0.$$

We denote by F the reaction force applied by the obstacle $B(0, r)$ on the fluid, it is defined as:

$$(33) \quad F = \int_{\partial\Omega} (\nabla u + \nabla u^\top - \pi \mathbb{I}) \cdot n d\sigma.$$

The following lemma provides us an equivalent definition of F :

Lemma A.1. *Let $R_0 \geq r$, there holds:*

$$F = \int_{\partial B(0, R_0)} [\nabla u - \pi \mathbb{I}] \cdot n d\sigma.$$

Proof. The aim is to prove that for arbitrary $W \in \mathbb{R}^3$:

$$F \cdot W = \int_{\partial B(0, R_0)} [(\nabla u - \pi \mathbb{I}) \cdot n] \cdot W d\sigma$$

Fix a vector-field $w \in \mathcal{C}_c^\infty(\mathbb{R}^3)$ such that $\operatorname{div} w = 0$, $w = W$ on $B(0, R_0)$, extend u by the value V on $B(0, r)$ and still denote u the extension for simplicity. After integration by

parts we obtain:

$$\begin{aligned}
 F \cdot W &= \int_{\mathbb{R}^3 \setminus B(0,r)} \nabla u : \nabla w + \nabla u : \nabla w^\top \\
 &= \int_{\mathbb{R}^3} \nabla u : \nabla w \\
 &= \int_{\mathbb{R}^3 \setminus B(0,R_0)} \nabla u : \nabla w \\
 &= \int_{\partial B(0,R_0)} [(\nabla u - \pi \mathbb{I}) \cdot n] \cdot W d\sigma
 \end{aligned}$$

As $\operatorname{div} u = 0$ and $w = W$ on $B(0,r)$. □

REFERENCES

- [1] G. Allaire. Homogenization of the Navier-Stokes equations in open sets perforated with tiny holes. I. Abstract framework, a volume distribution of holes. *Arch. Rational Mech. Anal.*, 113(3):209–259, 1990.
- [2] D. Cioranescu and F. Murat. Un terme étrange venu d’ailleurs. In *Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. II (Paris, 1979/1980)*, volume 60 of *Res. Notes in Math.*, pages 98–138, 389–390. Pitman, Boston, Mass.-London, 1982.
- [3] A. Yu. Beliaev and S. M. Kozlov. Darcy equation for random porous media. *Comm. Pure Appl. Math.*, 49(1):1–34, 1996.
- [4] E. Bernard, L. Desvillettes, F. Golse, and V. Ricci. A derivation of the vlasov-navier-stokes model for aerosol flows from kinetic theory. <https://hal-polytechnique.archives-ouvertes.fr/hal-01350730>, August 2016.
- [5] L. Boudin, L. Desvillettes, C. Grandmont, and A. Moussa. Global existence of solutions for the coupled Vlasov and Navier-Stokes equations. *Differential Integral Equations*, 22(11-12):1247–1271, 2009.
- [6] L. Desvillettes, F. Golse and V. Ricci. The mean field limit for solid particles in a Navier-Stokes flow. *J. Stat. Phys.* 131: 941-967, 2008.
- [7] G. P. Galdi. An introduction to the mathematical theory of the Navier-Stokes equations. Springer Monographs in Mathematics. Springer, New York, second edition, 2011.
- [8] K. Hamdache. Global existence and large time behaviour of solutions for the Vlasov-Stokes equations. *Japan J. Indust. Appl. Math.*, 15(1):51–74, 1998.
- [9] M. Hauray and P.-E. Jabin. Particle approximation of Vlasov equations with singular forces: propagation of chaos. *Ann. Sci. Éc. Norm. Supér. (4)*, 48(4):891–940, 2015.
- [10] E. Guazzelli and J. F. Morris A Physical Introduction To Suspension Dynamics. Cambridge Texts In Applied Mathematics. 2012
- [11] M. Hillairet. *On the homogenization of the Stokes problem in a perforated domain.* <https://hal.archives-ouvertes.fr/hal-01302560>, August 2016.
- [12] L. D. Landau and E. M. Lifshitz. *Fluid mechanics*. Translated from the Russian by J. B. Sykes and W. H. Reid. Course of Theoretical Physics, Vol. 6. Pergamon Press, London, 1959.
- [13] J.-P. Raymond. Stokes and Navier-Stokes equations with nonhomogeneous boundary conditions. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 24(6):921–951, 2007.

- [14] J. Rubinstein. On the macroscopic description of slow viscous flow past a random array of spheres. *J. Stat. Phys.*, 44(5-6):849–863, 1986.
- [15] C. Villani. *Optimal transport*, volume 338 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2009. Old and new.

INSTITUT MONTPELLIÉRAIN ALEXANDER GROTHENDIECK, UNIVERSITÉ DE MONTPELLIER, PLACE
EUGÈNE BATAILLON, 34095 MONTPELLIER CEDEX 5 FRANCE

E-mail address: amina.mecherbet@umontpellier.fr, matthieu.hillairet@umontpellier.fr